NORTH-HOLLAND

On Convergence of Nested Stationary Iterative Methods*

Zhi-Hao Cao

Department of Mathematics
Fudan University
Shanghai 200433,
People's Republic of China

Submitted by Hans Schneider

ABSTRACT

We study the convergence of nested stationary iterative methods according to Lanzkron, Rose, and Szyld. In contrast with their assumption that the coefficient matrix is monotone, our assumption on the coefficient matrix includes H-matrices.

1. INTRODUCTION

Lanzkron, Rose, and Szyld [3] analyzed the convergence of nested linear stationary iterative methods for the solution of linear systems of equations of the form

$$Ax = b, \tag{1}$$

where A is an $n \times n$ nonsingular matrix and x and b are vectors. They presented conditions on the splittings corresponding to the iterative methods to guarantee convergence for any number of inner iterations. These conditions imply that the coefficient matrix A must be a monotone one, i.e.,

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$A^{-1} \geq 0$ (A^{-1} is nonnegative). In this paper we analyze the convergence of the nested linear stationary iterative methods and present conditions on the splittings corresponding to the iterative methods to guarantee convergence for any number of inner iterations. It is not necessary for the coefficient matrix A in (1) to be a monotone matrix in order to satisfy the conditions presented in [3]. In particular, we shall show that an H-matrix A is a qualified candidate.

2. PREREQUISITES

In this paper we follow closely the notation and conception presented in [1], without explanation. We begin with the most general method of solution of (1) considered in [3], i.e. nested block iterations, which include the block Gauss-Seidel iterative method as a special case. We say that $A = M - N_1 - N_2$ is a composite splitting of A when M is nonsingular, and a convergent regular composite splitting if both $M_1 \equiv M - N_2$ and $A = M_1 - N_1$ are convergent regular splittings. Let $A = M - N_1 - N_2$ be partitioned into $q \times q$ blocks, and $M = F - G$ be block diagonal; then the nested block iteration was defined as follows.

ALGORITHM (Nested block iteration).

For $k = 0, 1, \ldots$
  $y_0 = y = \left[y^{(1)}, y^{(2)}, \ldots, y^{(q)}\right]^T = x_k$
  For $i = 1, 2, \ldots, q$
    For $j = 0, 1, \ldots, p_{i,k} - 1$
      $F_i y_{j+1}^{(i)} = (b + N_1 x_k + N_2 x_{k+1})^{(i)} + G_i y_j^{(i)}$
  $x_{k+1}^{(i)} = y_{p_{i,k}}^{(i)}$.

If $q = 1$ and $p_{i,k} = p$ for all k, then the iterative method is called a two-stage method.

Let $H = F^{-1}G$ be the iteration matrix of one step of the inner iteration (i.e. $M = F - G$); then the global iteration matrix of two-stage method is

$$T = (B - N_2)^{-1}(C + N_1), \tag{2}$$

where B and C are the unique matrices induced by the iteration matrix $R = H^p = B^{-1}C$ on the matrix $M = B - C$. The unique splitting induced by T on A is

$$A = M_T - N_T, \tag{3}$$

where

$$\begin{aligned} M_T &= B - N_2 = M(I-R)^{-1} - N_2, \\ N_T &= C + N_1 = M(I-R)^{-1}R + N_1. \end{aligned} \tag{4}$$

LEMMA 1 [4]. *Let $A = M - N$ be a weak regular splitting. Then A is monotone (i.e. $A^{-1} \geq 0$) if and only if $\rho(M^{-1}N) < 1$. In particular, let $T \geq 0$; then $I - T$ is monotone if and if $\rho(T) < 1$.*

Applying Lemma 1, the following convergence result was derived [3].

THEOREM 2. *Let $A = M - N_1 - N_2$ be a convergent regular composite splitting, and let $R \geq 0$, $\rho(R) < 1$. If the splitting $M = B - C$ such that $R = B^{-1}C$ is weak regular, then the iterative method defined by*

$$A = M_T - N_T,$$

$$M_T = M(I-R)^{-1} - N_2, \qquad N_T = M(I-R)^{-1}R + N_1$$

is convergent. Moreover, $A = M_T - N_T$ is weak regular splitting.

The convergence of the two-stage method is a consequence of the above theorem.

THEOREM 3. *Let $A = M - N_1 - N_2$ be a convergent regular composite splitting, and let p be a positive integer. If $M = F - G$ is a weak regular splitting, then the two-stage iterative method is convergent, and its induced splitting is weak regular.*

REMARK 1. The assumption that $A = M - N_1 - N_2$ is a convergent regular composite splitting implies that A is monotone (cf. Lemma 1).

3. MAIN RESULT

We first consider the convergence of the two-stage methods.

THEOREM 4. *Let $A = M - N_1 - N_2$ be a composite splitting such that $\widetilde{A} = M - |N_1| - |N_2|$ is a convergent regular composite splitting and $M = B - C$ is a weak regular splitting. Then the iterative method defined by*

$$A = M_T - N_T,$$

$$\begin{aligned} M_T &= B - N_2 = M(I-R)^{-1} - N_2, \\ N_T &= C + N_1 = M(I-R)^{-1}R + N_1, \end{aligned} \tag{5}$$

where $R = B^{-1}C$, *is convergent.*

Proof. Since the splitting $M = B - C$ is weak regular and M is a monotone matrix, we have (see Lemma 1)

$$B^{-1} \geq 0, \qquad R \equiv B^{-1}C \geq 0, \quad \text{and} \quad \rho(R) < 1.$$

Let

$$\widetilde{A} = M_{\tilde{T}} - N_{\tilde{T}},$$

$$M_{\tilde{T}} = M(I-R)^{-1} - |N_2| \quad \text{and} \quad N_{\tilde{T}} = M(I-R)^{-1}R + |N_1|. \tag{6}$$

Theorem 2 implies that (6) is a weak regular splitting and

$$\widetilde{T} \equiv M_{\tilde{T}}^{-1}N_{\tilde{T}} \geq 0, \rho(\widetilde{T}) < 1 \quad \text{and} \quad (I - \widetilde{T})^{-1} \geq 0.$$

Clearly,

$$|B^{-1}N_2| \leq (I-R)^{-1}|B^{-1}N_2| \leq (I-R)^{-1}B^{-1}|N_2| = M^{-1}|N_2|. \tag{7}$$

This implies (cf. [5, 6])

$$\rho(B^{-1}N_2) \leq \rho(|B^{-1}N_2|) \leq \rho(M^{-1}|N_2|) < 1. \tag{8}$$

From (8) we know that the iteration matrix T of the iterative method defined by (5) is well defined:

$$T \equiv (B - N_2)^{-1}(C + N_1) = (I - B^{-1}N_2)^{-1}(R + B^{-1}N_1),$$

and it follows that

$$\begin{aligned} |T| &\leq |(I - B^{-1}N_2)^{-1}|(R + |B^{-1}N_1|) \\ &\leq (I - B^{-1}|N_2|)^{-1}(R + B^{-1}|N_1|) = \widetilde{T}. \end{aligned}$$

Therefore, we have

$$\rho(T) \leq \rho(\widetilde{T}) < 1.$$

This finishes the proof of the theorem. ∎

The matrix A in Theorem 4 can be an H-matrix, as we shall show (Theorem 6). Let $\mathcal{M}(A)$ denote the comparison matrix of A.

THEOREM 5. *Let $A = M - N_1 - N_2$ be a composite splitting such that $\widehat{A} = \mathcal{M}(M) - |N_1| - |N_2|$ is a convergent regular composite splitting. Let $M = B - C$ be a convergent weak regular splitting with $R = B^{-1}C$. Then the iterative method defined by* (5) *is convergent.*

Proof. Since $M = B - C$ is a convergent weak regular splitting. Lemma 1 implies $M^{-1} \geq 0$. It is easy to see that $\mathcal{M}(M)^{-1} \geq 0$. Therefore, M is a monotone H-matrix, and it follows (cf. [2]) that

$$M^{-1} = |M^{-1}| \leq \mathcal{M}(M)^{-1}.$$

Thus, we have

$$\rho\big(M^{-1}|N_2|\big) \leq \rho\big(\mathcal{M}(M)^{-1}|N_2|\big) < 1, \tag{9}$$

and $M - |N_2|$ is a convergent regular splitting. Lemma 1 implies

$$(M - |N_2|)^{-1} \geq 0.$$

Let

$$Q = (M - |N_2|)^{-1}|N_1|;$$

then we have

$$\begin{aligned} Q &= \big(I - M^{-1}|N_2|\big)^{-1} M^{-1}|N_1| \\ &\leq \big[I - \mathcal{M}(M)^{-1}|N_2|\big]^{-1}\mathcal{M}(M)^{-1}|N_1| \\ &= [\mathcal{M}(M) - |N_2|]^{-1}|N_1|. \end{aligned} \tag{10}$$

This implies $\rho(Q) < 1$, i.e., $(M - |N_2|) - |N_1|$ is a convergent regular splitting.

Combining this fact with (9) it follows that

$$\widetilde{A} = M - |N_1| - |N_2|$$

is a convergent regular composite splitting, and the theorem follows from Theorem 4. ∎

THEOREM 6. *Let $A = M - N_1 - N_2$ be a composite splitting such that $\widehat{A} = \mathcal{M}(M) - |N_1| - |N_2|$ is an M-matrix. Let $M = B - C$ be a convergent weak regular splitting with $R = B^{-1}C$. Then the iterative method defined by* (5) *is convergent.*

Proof. That $\widehat{A}$ is an M-matrix and

$$\mathcal{M}(M) \geq \widehat{A}$$

imply (see [2]) that $\mathcal{M}(M)$ is an M-matrix. Since

$$\widehat{A} = \mathcal{M}(M) - (|N_1| + |N_2|)$$

is a regular splitting of the M-matrix $\widehat{A}$, Lemma 1 implies

$$\rho(\mathcal{M}(M)^{-1}(|N_1| + |N_2|)) < 1,$$

and it follows that

$$\rho\big(\mathcal{M}(M)^{-1}|N_2|\big) \leq \rho\big(\mathcal{M}(M)^{-1}(|N_1| + |N_2|)\big) < 1,$$

which implies

$$M_1 \equiv \mathcal{M}(M) - |N_2|$$

is a convergent regular splitting of the matrix M_1. Therefore, $M_1^{-1} \geq 0$, and

$$\widehat{A} = M_1 - |N_1|$$

is a convergent regular splitting.

Thus, we have shown that $\widehat{A} = \mathcal{M}(M) - |N_1| - |N_2|$ is a convergent regular composite splitting, and the theorem follows from Theorem 5. ■

REMARK 2. From the assumption on A in Theorem 6, we have

$$\mathcal{M}(A) = \mathcal{M}(M - N_1 - N_2) \geq \mathcal{M}(M) - |N_1| - |N_2| = \widehat{A},$$

which implies $\mathcal{M}(A)$ must be an M-matrix, i.e., A must be an H-matrix.

A particular case of assumption on A in Theorem 6 is that A is an H-matrix and $A = M - N_1 - N_2$ is a composite splitting such that $\mathcal{M}(A) = \mathcal{M}(M) - |N_1| - |N_2|$.

The convergence in three cases of the two-stage method is a consequence of the above three theorems, respectively.

THEOREM 7. *Let $A = M - N_1 - N_2$ be composite splitting satisfying conditions in Theorem 4 or Theorem 5 (or, when A is an H-matrix, in Theorem 6), and let p be a positive integer. If $M = F - G$ is a convergent weak regular splitting, then the two-stage iterative method is convergent.*

Proof. In all three cases it suffices to show that $M = B - C$ is a weak regular splitting with

$$B = M\left[I - (F^{-1}G)^p\right]^{-1} \quad \text{and} \quad C = M\left[I - (F^{-1}G)^p\right]^{-1}(F^{-1}G)^p.$$

Since $F - G$ is a weak regular splitting of the monotone matrix M, we have $F^{-1} \geq 0$, $F^{-1}G \geq 0$, and $\rho(F^{-1}G) < 1$. Therefore,

$$\begin{aligned} B^{-1} &= \left[I - (F^{-1}G)^p\right]M^{-1} = \left[I - (F^{-1}G)^p\right](I - F^{-1}G)^{-1}F^{-1} \\ &= \sum_{j=0}^{p-1}(F^{-1}G)^j F^{-1} \geq 0 \end{aligned}$$

and $B^{-1}C = (F^{-1}G)^p \geq 0$.

These imply that the splitting $M = B - C$ is weak regular, and the proof of the theorem is finished. ■

If $p_{i,k} = p_k$ for all i in the Algorithm (nested block iteration), then the algorithm is called dynamic nested iteration [3], and it can be viewed as the concatenation of different nested methods, each with a different iteration matrix T_{p_k}, which is a two-stage iterations matrix with number of inner iterations equal to p_k. Thus, for r outer iterations, the global iterations matrix for the dynamic nested iteration is the product $T_{p_r}T_{p_{r-1}}\cdots T_{p_1}$.

THEOREM 8. *Let $M = F - G$ be a convergent regular splitting. If $A = M - N_1 - N_2$ is a composite splitting satisfying conditions in Theorem 4 or Theorem 5 (or, when A is an H-matrix, in Theorem 6), then the dynamic nested iterative method with r iterations is convergent for every r.*

Proof. In all three cases we know that $\widetilde{A} = M - |N_1| - |N_2|$ is a convergent regular composite splitting.

As in the proof of Theorem 4, let

$$\widetilde{A} = M_{\tilde{T}_{p_k}} - N_{\tilde{T}_{p_k}}$$

with

$$\begin{aligned} M_{\tilde{T}_{p_k}} &= M(I - R_{p_k})^{-1} - |N_2| = B_{p_k} - |N_2|, \\ N_{\tilde{T}_{p_k}} &= M(I - R_{p_k})^{-1}R_{p_k} + |N_1| = C_{p_k} + |N_1|, \end{aligned}$$

$k = 1, 2, \ldots, r$, where $R_{p_k} = (F^{-1}G)^{p_k}$, which induces unique matrices B_{p_k}

and C_{p_k} (i.e. $B_{p_k}^{-1}C_{p_k} = R_{p_k}$) on the matrix $M = F - G = B_{p_k} - C_{p_k}$:

$$B_{p_k} = F(I - F^{-1}G)(I - R_{p_k})^{-1}$$

and

$$C_{p_k} = F(I - F^{-1}G)(I - R_{p_k})^{-1}R_{p_k}.$$

Denote by $\widetilde{T}_{p_k}$ the corresponding two-stage iteration matrix:

$$\widetilde{T}_{p_k} = M_{\tilde{T}_{p_k}}^{-1} N_{\tilde{T}_{p_k}}.$$

From the proof of Theorem 7 we know that $M = B_{p_k} - C_{p_k}$ is a weak regular splitting, i.e.,

$$B_{p_k}^{-1} \geq 0 \quad \text{and} \quad B_{p_k}^{-1}C_{p_k} \geq 0.$$

Since $M^{-1}|N_2|$ is convergent and

$$B_{p_k}^{-1}|N_2| \leq (I - R_{p_k})^{-1}B_{p_k}^{-1}|N_2| = M^{-1}|N_2|,$$

which imply that $\rho(B_{p_k}^{-1}|N_2|) < 1$ and

$$\begin{aligned} \widetilde{T}_{p_k} &= (B_{p_k} - |N_2|)^{-1}(C_{p_k} + |N_1|) \\ &= \left(I - B_{p_k}^{-1}|N_2|\right)^{-1}\left(B_{p_k}^{-1}C_{p_k} + B_{p_k}^{-1}|N_1|\right) \geq 0. \end{aligned}$$

It has been proved that (see [3, Theorem 6.4])

$$\rho(\widetilde{T}_{p_r}\widetilde{T}_{p_{r-1}} \cdots \widetilde{T}_{p_1}) \leq [\rho(\widetilde{T}_1)]^r < 1.$$

As in the proof of Theorem 4, for a two-stage iteration matrix

$$T_{p_k} = (B_{p_k} - N_2)^{-1}(C_{p_k} + N_1), \qquad k = 1, 2, \ldots, r,$$

we have

$$|T_{p_k}| \leq (B_{p_k} - |N_2|)^{-1}(C_{p_k} + |N_1|) = \widetilde{T}_{p_k}, \qquad k = 1, 2, \ldots, r,$$

which imply

$$|T_{p_r}T_{p_{r-1}} \cdots T_{p_1}| \leq \widetilde{T}_{p_r}\widetilde{T}_{p_{r-1}} \cdots \widetilde{T}_{p_1}.$$

Thus, it follows that

$$\rho(T_{p_r}T_{p_{r-1}} \cdots T_{p_1}) \leq \rho(\widetilde{T}_{p_r}\widetilde{T}_{p_{r-1}} \cdots \widetilde{T}_{p_1}) < 1,$$

which completes the proof of the theorem. ∎

We now consider the most general method, i.e., nested block iteration. Since $M = F - G = \operatorname{diag}(M_1, \ldots, M_q)$ is block diagonal, $F = \operatorname{diag}(F_1, \ldots, F_q)$ and $G = \operatorname{diag}(G_1, \ldots, G_q)$. Let $R_i = F_i^{-1}G_i$, $i = 1, \ldots, q$; then $R = F^{-1}G = \operatorname{diag}(R_1, \ldots, R_q)$. Let $\mathcal{P}_k = \{p_{1,k}, p_{2,k}, \ldots, p_{q,k}\}$, and define

$$\begin{aligned}\mathcal{B}(\mathcal{P}_k) &= \operatorname{diag}\big(M_1(I_{s_1} - R_1^{p_{1,k}})^{-1}, \ldots, M_q(I_{s_q} - R_q^{p_{q,k}})^{-1}\big),\\ \mathcal{R}(\mathcal{P}_k) &= \operatorname{diag}(R_1^{p_{1,k}}, \ldots, R_q^{p_{q,k}}).\end{aligned}$$

From this it follows that

$$\mathcal{B}(\mathcal{P}_k) = M[I - \mathcal{R}(\mathcal{P}_k)]^{-1}.$$

It is easy to see that the iteration matrix for the nested block iteration is given by

$$\mathcal{H}(\mathcal{P}_k) = \big[I - \mathcal{B}(\mathcal{P}_k)^{-1}N_2\big]^{-1}\big[\mathcal{R}(\mathcal{P}_k) + \mathcal{B}(\mathcal{P}_k)^{-1}N_1\big],$$

which induces the unique splitting on A:

$$A = M_H(\mathcal{P}_k) - N_H(\mathcal{P}_k)$$

with $\mathcal{H}(\mathcal{P}_k) = M_H(\mathcal{P}_k)^{-1}N_H(\mathcal{P}_k)$, where

$$M_H(\mathcal{P}_k) = \mathcal{B}(\mathcal{P}_k) - N_2, \qquad N_H(\mathcal{P}_k) = \mathcal{B}(\mathcal{P}_k)\mathcal{R}(\mathcal{P}_k) + N_1.$$

If $p_{i,k} = p_i, i = 1, \ldots, q$, for all k, then we have the following result.

THEOREM 9. *If $M = F - G$ is a weak regular splitting and $A = M - N_1 - N_2$ is a composite splitting such that $\tilde{A} = M - |N_1| - |N_2|$ is a convergent regular composite splitting, then $\rho(\mathcal{H}(\mathcal{P})) < 1$, i.e. the nested block iterative method in this case is convergent.*

Proof. Obviously, we only need to show that $M = \mathcal{B}(\mathcal{P}) - \mathcal{B}(\mathcal{P})\mathcal{R}(\mathcal{P})$ is a weak regular splitting, and the theorem will follow from Theorem 4. Clearly, $\mathcal{R}(\mathcal{P})$ is nonnegative, $\rho(\mathcal{R}(\mathcal{P})) < 1$, and

$$\begin{aligned}\mathcal{B}(\mathcal{P})^{-1} &= [I - \mathcal{R}(\mathcal{P})]M^{-1}\\ &= \operatorname{diag}\big((I_{s_1} - R_1^{p_1})M_1^{-1}, \ldots, (I_{s_q} - R_q^{p_q})M_q^{-1}\big).\end{aligned}$$

Since

$$\begin{aligned}(I_{s_i} - R_i^{p_i})M_i^{-1} &= (I_{s_i} - R_i^{p_i})(I - R_i)^{-1}F_i^{-1} \\ &= \sum_{j=0}^{p_i-1} R_i^j F_i^{-1} \geq 0, \qquad i = 1, \ldots, q.\end{aligned}$$

It follows that $\mathcal{B}(\mathcal{P})^{-1} \geq 0$ and the proof of the theorem is finished. ∎

Apply Theorems 5 and 6 we can easily prove the following theorem.

THEOREM 10. *Let $A = M - N_1 - N_2$ be composite splitting satisfying the conditions in Theorem 5 or, when A is an H-matrix, in Theorem 6. If $M = F - G$ is a convergent weak regular splitting, then $\rho(\mathcal{H}(\mathcal{P})) < 1$, i.e., the nested black iterative method in this case is convergent.*

For the general nested block method, the global iteration matrix for r outer iterations is the product $\mathcal{H}(\mathcal{P}_r) \cdots \mathcal{H}(\mathcal{P}_1)$.

THEOREM 11. *Let $M = F - G$ be a convergent regular splitting. If $A = M - N_1 - N_2$ is a composite splitting satisfying conditions in Theorem 4 or Theorem 5(or, when A is an H-matrix, in Theorem 6), then the nested block iterative method is convergent.*

Proof. The proof is analogous to that of Theorem 8. Let

$$\widetilde{\mathcal{H}}(\mathcal{P}_k) = \left[I - \mathcal{B}(\mathcal{P}_k)^{-1}|N_2|\right]^{-1}\left[\mathcal{R}(\mathcal{P}_k) + \mathcal{B}(\mathcal{P}_k)^{-1}|N_1|\right].$$

It is easy to show that

$$\widetilde{\mathcal{H}}(\mathcal{P}_k) \geq 0$$

and

$$\rho(\widetilde{\mathcal{H}}(\mathcal{P}_r)\widetilde{\mathcal{H}}(\mathcal{P}_{r-1}) \cdots \widetilde{\mathcal{H}}(\mathcal{P}_1)) < 1.$$

Then we have

$$\begin{aligned}|\mathcal{H}(\mathcal{P}_k)| &\leq \left|(I - \mathcal{B}(\mathcal{P}_k)^{-1}N_2)^{-1}\right|\left|\mathcal{R}(\mathcal{P}_k) + \mathcal{B}(\mathcal{P}_k)^{-1}N_1\right| \\ &\leq \left[I - \mathcal{B}(\mathcal{P}_k)^{-1}|N_2|\right]^{-1}\left[\mathcal{R}(\mathcal{P}_k) + \mathcal{B}(\mathcal{P}_k)^{-1}|N_1|\right] \\ &= \widetilde{\mathcal{H}}(\mathcal{P}_k),\end{aligned}$$

which imply

$$|\mathcal{H}(\mathcal{P}_r)\mathcal{H}(\mathcal{P}_{r-1}) \cdots \mathcal{H}(\mathcal{P}_1)| \leq \widetilde{\mathcal{H}}(\mathcal{P}_r)\widetilde{\mathcal{H}}(\mathcal{P}_{r-1}) \cdots \widetilde{\mathcal{H}}(\mathcal{P}_1).$$

Thus, we get

$$\rho(\mathcal{H}(\mathcal{P}_r)\mathcal{H}(\mathcal{P}_{r-1})\cdots\mathcal{H}(\mathcal{P}_1)) \leq \rho(\widetilde{\mathcal{H}}(\mathcal{P}_r)\widetilde{\mathcal{H}}(\mathcal{P}_{r-1})\cdots\widetilde{\mathcal{H}}(\mathcal{P}_1)) < 1.$$

This finishes the proof of the theorem. ∎

As was noticed in [3], the theory we are developing can be extended to the case of recursive inner iterations, which were called nested iterative method [3]. Consider the solution of (1) by a two-stage method with the outer iteration defined by the composite splitting $A = M - N_1 - N_2$ which satisfies the conditions in Theorem 4 or 5 (or Theorem 6 when A is an H-matrix). Then instead of solving the system $Mx = f(x_k, x_{k+1}, b)$ by an iterative method, it is solved by a two-stage method (replace A by M, and b by $b + N_1x_k + N_2x_{k+1}$) and so on. This implies the specification of a new composite splitting with iteration matrix $F^{-1}G$, $M = F - G$, and number of iterations equal to p at each level. One could interpret $M = F - G$ as (5). Since the split matrix in each level except the outermost [i.e. the matrix A in (1)] is monotone, each composite splitting except the outermost can be taken as a convergent regular one, and the resulting splitting (5) induced by the two-stage iteration matrix is a weak regular splitting (cf. Theorem 2).

By induction on the number of levels and applying Theorem 7 and Theorem 2 we can easily prove the following result.

THEOREM 12. *Let $A = M - N_1 - N_2$ be a composite splitting satisfying the conditions in Theorem 4 or Theorem 5(or Theorem 6 when A is an H-matrix). Let $M = F - G$ be a convergent weak regular splitting. Moreover, at each level except the outermost, let the refined $A = M - N_1 - N_2$ be a convergent regular composite splitting, and at the innermost level let $M = F - G$ be a weak regular splitting. Then the corresponding nested iterative method is convergent.*

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